COMP 232 Mathematics for Computer Science
Winter 2014
Midterm Exam

Name: ______________________ Total Points:

ID: ______________________ ________ / 59

Instructions. This is a closed book exam. The only allowed tool is an ENCS approved calculator. Provide all answers in this booklet. Use pen, not pencil. Do not detach any pages from this exam!

(3$^{\text{pts}}_{\text{ea.}}$) 1. We know that $\{\wedge, \neg\}$ forms a functionally complete set of operators, meaning that any other operator can be defined in terms of $\{\wedge, \neg\}$ only, for example

9 pts

$$\begin{aligned} p \vee q &=_{\mathsf{def}} \neg(\neg p \wedge \neg q) \\ p \to q &=_{\mathsf{def}} \neg(p \wedge \neg q) \end{aligned}$$

The Shaffer stroke $\uparrow$ is a binary operator that has the following truth table:

p	q	$p \uparrow q$
T	T	F
T	F	T
F	T	T
F	F	T

Show that the Shaffer stroke by itself is functionally complete, by defining in the space below, the following operators by using the Shaffer stroke only:

(a) $\neg p$

(b) $p \wedge q$

(c) $p \vee q$

9 pts

(2$^{\text{pts}}_{\text{ea.}}$) 2. For each of the following propositional sentences, state whether or not it is a tautology.

You get +2 points for each correct answer, -2 points for each wrong answer, and 0 points for "don't know." However, the total for this question will not be less that 0.

10 pts

(a) $((p \vee q) \wedge (q \vee r)) \leftrightarrow (q \vee r)$

☐ Not tautology ☐ Tautology ☐ Don't know!

(b) $(p \to q) \to (q \to p)$

☐ Tautology ☐ Not a tautology ☐ Don't know!

(c) $((p \to q) \wedge (q \to p)) \leftrightarrow (p \leftrightarrow q)$

☐ Tautology ☐ Not a tautology ☐ Don't know!

(d) $((p \to q) \wedge (\neg r \to \neg q)) \to (p \to (q \wedge r))$

☐ Tautology ☐ Not a tautology ☐ Don't know!

(e) $(p \to (q \to r)) \leftrightarrow ((p \to q) \to r)$

☐ Not a tautology ☐ Tautology ☐ Don't know!

10 pts

($6^{\text{pts}}_{\text{ea.}}$) 3. Here you are to prove propositional equivalences using the laws in the handout.

12 pts

(a) In the table below, construct a proof of the equivalence

$$(r \vee p) \rightarrow (r \vee q) \equiv r \vee (p \rightarrow q)$$

Step	Law applied

(b) In the table below, construct a proof of the equivalence

$$\neg(p \rightarrow q) \rightarrow p \equiv True$$

Step	Law applied

12 pts

($2^{\text{pts}}_{\text{ea.}}$) 4. Let the universe of discourse be $\mathbb{Z}^+$, the set $\{1, 2, 3, \ldots\}$ of positive integers. Let $P(x, y, z)$ denote the statement "z is a multiple of $x+y$" or, equivalently, "There is a positive integer q, such that $z = (x + y)q$"

8 pts

What is the truth value of each of the following? You get +2 points for each correct answer, -2 points for each wrong answer, and 0 points for "don't know." However, the total for this question will not be less that 0.

(a) $\forall x \exists y \exists z P(x, y, z)$

☐ False ☐ True ☐ Don't know!

(b) $\forall y \forall z \exists x P(x, y, z)$

☐ False ☐ True ☐ Don't know!

(c) $\forall x \forall y \exists z P(x, y, z)$

☐ False ☐ True ☐ Don't know!

(d) $\forall z \exists x \exists y P(x, y, z)$

☐ False ☐ True ☐ Don't know!

8 pts

(4pts) 5. The negation of the statement $\forall x \neg \forall y \exists z (P(x,z) \wedge Q(z,y))$ is

4 pts

- ☐ $\exists x \forall y \exists z (P(x,z) \wedge Q(z,y))$
- ☐ $\forall x \forall y \exists z (\neg P(x,z) \wedge \neg Q(z,y))$
- ☐ $\forall x \exists y \forall z (\neg P(x,z) \vee \neg Q(z,y))$
- ☐ $\forall x \exists y \forall z (\neg P(x,z) \wedge \neg Q(z,y))$
- ☐ $\exists x \exists y \forall z (P(x,z) \vee \neg Q(z,y))$

(4pts) 6. The proposition $(p \leftrightarrow r) \rightarrow (q \leftrightarrow r)$ is equivalent to

4 pts

- ☐ $((\neg p \vee r) \wedge (p \vee \neg r)) \vee \neg((\neg q \vee r) \wedge (q \vee \neg r))$
- ☐ $((\neg p \vee r) \wedge (p \vee \neg r)) \wedge ((\neg q \vee r) \wedge (q \vee \neg r))$
- ☐ $\neg((\neg p \vee r) \wedge (p \vee \neg r)) \vee ((\neg q \vee r) \wedge (q \vee \neg r))$
- ☐ $((\neg p \vee r) \wedge (p \vee \neg r)) \vee ((\neg q \vee r) \wedge (q \vee \neg r))$

(4pts) 7. Which of the following statements is the contrapositive of the statement "You win the game if you know the rules but are not overconfident."

4 pts

- ☐ "If you lose the game then you don't know the rules or you are overconfident."
- ☐ "If you don't know the rules and are overconfident then you win the game."
- ☐ "If you don't know the rules or are overconfident then you lose the game."
- ☐ "A necessary condition that you know the rules or you are not overconfident is that you win the game."
- ☐ "A sufficient condition that you win the game is that you know the rules or you are not overconfident."

12 pts

(4[pts]) 8. Let $P(x, y)$ mean "x loves y." Which of the following expresses the proposition "Everybody loves somebody who doesn't love anybody." 4 pts

- ☐ $\forall x \exists y\ (P(x, y) \wedge \forall z(\neg P(y, z)))$
- ☐ $\forall x \exists y \forall z\ (P(x, y) \wedge \neg P(z, y))$.
- ☐ $\forall x \forall y\ (P(x, y) \wedge \exists z(\neg P(y, z)))$
- ☐ $\forall x \exists y \exists z\ (P(x, y) \wedge \neg P(z, y))$.

(4[pts]) 9. Let the Universe of Discourse be $\mathbb{Z}$. Consider the assertion 4 pts

$$\exists x \Big(P(x) \wedge Q(x) \Big) \equiv \Big(\exists x \big(P(x) \big) \Big) \wedge \Big(\exists x \big(Q(x) \big) \Big)$$

Which of the following statements correctly describes the assertion?

- ☐ The assertion is false. As a counterexample, let $P(x)$ mean "x is divisible by 6," and $Q(x)$ mean "x is divisible by 3."
- ☐ The assertion is false. For a counterexample, let $P(x)$ mean '" $x < 0$," and $Q(x)$ mean "$x \geq 0$."
- ☐ The assertion is false. For a counterexample, let $P(x)$ mean '"$x < 0$ is a square" and $Q(x)$ mean "x is odd."
- ☐ The assertion is true. The proof follows from the distributive laws for $\wedge$
- ☐ The assertion is true. To see why, let $P(x)$ mean "x is divisible by 6," and $Q(x)$ mean "x is divisible by 3." If $x = 6$, then x is divisible by both 3 and 6, so both side of the equivalence have the same truth value for this x.

—... End of Exam ...—

8 pts